Friedrich-Alexander-Universität
Naturwissenschaftliche Fakultät

Interactive Formative Assessments in Multivariate Analysis

1st Northern e-Assessment Meeting, Trondheim

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What will be shown?

Focus

- Visualization aspects and 3D-STACK
  - coordinate transformation,
  - calculus of function with two variable,
  - visualization of vector fields in 3D.
- Documentation
- Adaptive tasks

ERASMUS+ Interactive Digital Assessment in Mathematics

What to teach?

Overview of competencies

- Thinking mathematically
- Reasoning mathematically
- Posing and solving mathematical problems
- **Modelling mathematically**
- **Representing mathematical entities**
- **Handling mathematical symbols and formalism**
- Communicating in, with, and about mathematics
- **Making use of aids and tools**

Based on **A Framework for Mathematics Curricula in Engineering Education**, SEFI (2013)

Task catalogue

- Plug-and-play Moodle tasks
- Extensive documentation
  - Aim of task
  - Brief description
  - Preview screen captures
  - Step-by-step breakdown of code
  - Detailed specific feedback

Transformation Theorem

Theorem

Given two sets $G \subset \mathbb{R}^n$ *and* $H \subset \mathbb{R}^n$ *im* $\mathbb{R}^n$ *and a one-to-one mapping* $T : H \to G$

$$T(\mathbf{u}) := \mathbf{x}(\mathbf{u}).$$

T *is continuously differentiable and* $\det(J_T(u, v, w)) \neq 0$ *on* H*.*

Then

$$\int_G f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_H f(\mathbf{x}(\mathbf{u}))\,\left|\det J_T(\mathbf{u})\right|\,\mathrm{d}\mathbf{u}.$$

Polar Coordinates

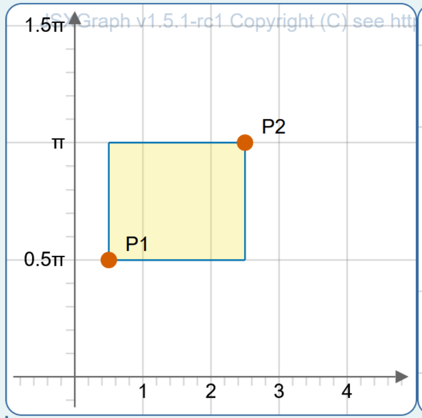
$$T : [0, \infty) \times [0, 2\pi) \to \mathbb{R}^2 \text{ with } \begin{pmatrix} r \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} r\cos(\phi) \\ r\sin(\phi) \end{pmatrix}$$

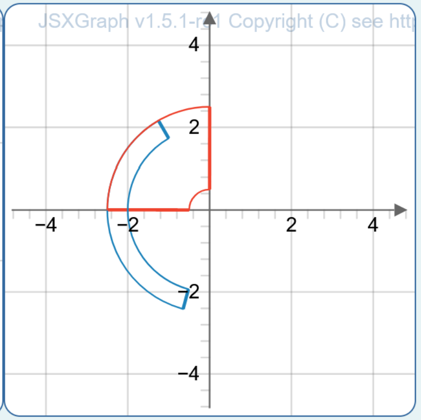
Tool zum Nachbessern der Frage | Es fehlen Tests oder Varianten.

Given is a 2D area with polar geometry. It is defined by the intervals for each of the polar coordinates r and ϕ. Here r is the radial coordinate and ϕ is the angle starting at the x-axis oriented counterclockwise with $\phi \in [0, 2\pi]$.

Reconstruct the intervals that define the given area by matching the areas using the cartesian coordinate system.

Write the interval in the form $r \in$ `[r1,r2]` and $\phi \in$ `[phi1,phi2]`, e.g. `[1/2,2]` and `[1/2*pi,2*pi]`.





$r \in$ ________

$\phi \in$ ________

Polar Coordinates

$$T : [0, \infty) \times [0, 2\pi) \to \mathbb{R}^2 \text{ with } \begin{pmatrix} r \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} r\cos(\phi) \\ r\sin(\phi) \end{pmatrix}$$

Idea

- Student knows the transformation from cartesian to polar coordinates (Handling mathematical symbols and formalism)
- Student can think of an area that is transformed to the given area in polar coordinates and check graphically (represent mathematical entities, posing and solving mathematical problems, making use of aids and tools)

Alternatives

1. points `P1` und `P2` binded to the students answer (matching)
2. intervals has to entered by the students (algebraic)

Spherical Coordinates

Transformation

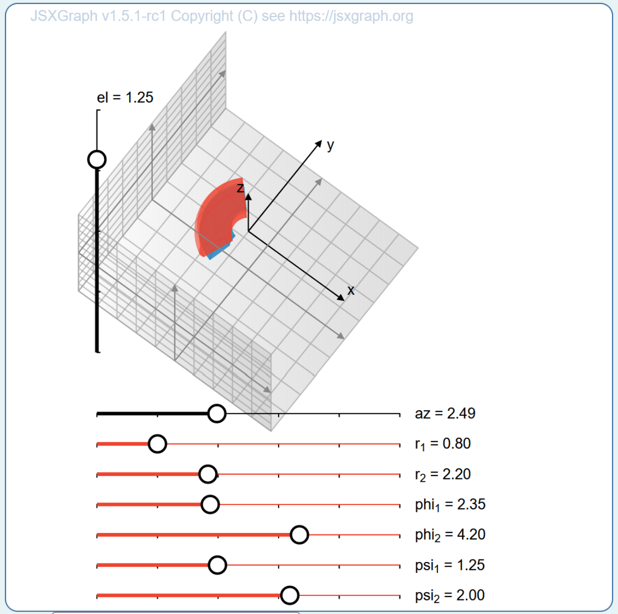
$T : \mathbb{R}_0^+ \times [0, 2\pi) \times [0, \pi] \to \mathbb{R}^3$ with

$$\begin{pmatrix} r \\ \phi \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} r\cos(\phi)\sin(\psi) \\ r\sin(\phi)\sin(\psi) \\ r\cos(\psi) \end{pmatrix}$$

Spherical Coordinates

Given is a 3D volume with spherical geometry. It is defined by the intervals for each of the spherical coordinates r, ϕ and ψ. Here r is the radial coordinate and ϕ is the azimuthal angle starting at the x-axis oriented counterclockwise with $\phi \in [0, 2\pi]$. Lastly, ψ is the polar angle measured from the z-axis with $\psi \in [0, \pi]$. Tool zum Nachbessern der Frage | Es fehlen Tests oder Varianten.

Reconstruct the intervals that define the given volume.



$r_1 =$ 0.8

$r_2 =$ 2.2

$\phi_1 =$ 2.35

$\phi_2 =$ 4.2

$\psi_1 =$ 1.25

$\psi_2 =$

Spherical Coordinates

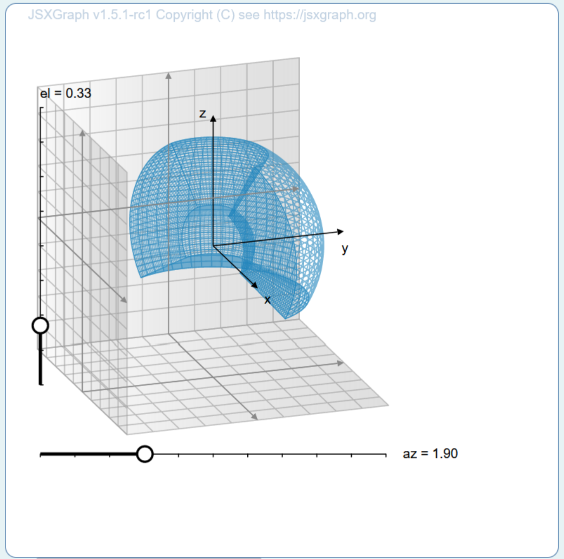
Tool zum Nachbessern der Frage | Es fehlen Tests oder Varianten

Given is a 3D volume with spherical geometry. It is defined by the intervals for each of the spherical coordinates r, ϕ and ψ. Here r is the radial coordinate and ϕ is the azimuthal angle starting at the x-axis oriented counterclockwise with $\phi \in [0, 2\pi]$. Lastly, ψ is the polar angle measured from the z-axis with $\psi \in [0, \pi]$.

Reconstruct the intervals that define the given volume.

Write the interval in the form $r \in$`[r1,r2]` and $\phi \in$`[phi1,phi2]`, e.g. `[1/2,2]` and `[1/2*pi,2*pi]`.

The generated radii are of the form $n/2, \in \mathbb{N}$ and the generated angles of the form $q \cdot \pi, q \in \mathbb{Q}$.



$r \in$ [1,4]

Ihre letzte Antwort wurde folgendermaßen interpretiert:

$$[1, 4]$$

$\phi \in$ [pi/6,7/6*pi]

Ihre letzte Antwort wurde folgendermaßen interpretiert:

$$\left[\frac{\pi}{6}, \frac{7}{6} \cdot \pi\right]$$

$\psi \in$ [pi/3,2/3*pi]

Spherical Coordinates

Your answer is partially correct.

The value you gave for r_1 is not correct.

Nice, you found the correct value for r_2! Good job!

Check whether you did anything different here than for r_1 and try again.

Correct answer, well done.

Nice, you found the correct value for ϕ_1! Good job!

Nice, you found the correct value for ϕ_2! Good job!

Perfect! You got both values of ϕ right!

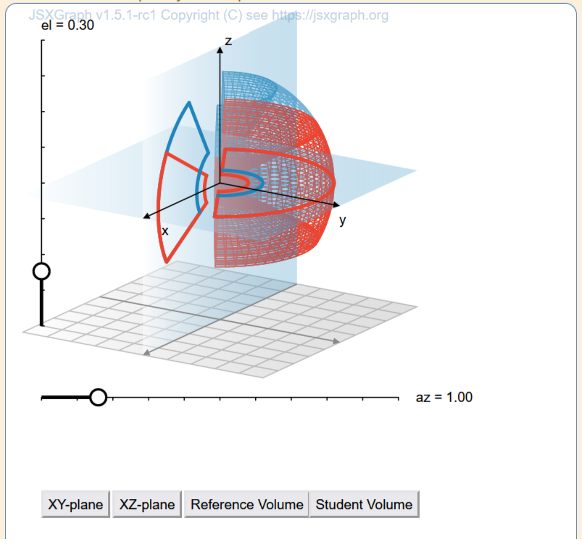
Your answer is partially correct.

The value you gave for ψ_1 is not correct.

Nice, you found the correct value for ψ_2! Good job!

Check whether you did anything different here than for ψ_1 and try again.

You can now compare your response to the reference volume. Your solution is displayed in orange. In addition, you can see the cross sections in the $x-y$-plane and $x-z$-plane. Note, that you can deactivate the visualizations using the button.



General Case

General setting

Given box $H = [u_1, u_2] \times [v_1, v_2] \times [w_1, w_2]$ and a transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ with

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} T_1(u, v, w) \\ T_2(u, v, w) \\ T_3(u, v, w) \end{pmatrix}$$

To display the transformed set $M = T([u_1, u_2] \times [v_1, v_2] \times [w_1, w_2])$ one needs six surfaces:

$$\begin{aligned} S_1(v, w) &= T(u_1, v, w) & S_3(u, w) &= T(u, v_1, w) & S_5(u, v) &= T(u, v, w_1) \\ S_2(v, w) &= T(u_2, v, w) & S_4(u, w) &= T(u, v_2, w) & S_6(u, v) &= T(u, v, w_2) \end{aligned}$$

Example

3D Transform

$$\begin{aligned} T_1(u, v, w) &= (3 + u\cos(v))\cos(w) \\ T_2(u, v, w) &= (3 + u\cos(v))\sin(w) \\ T_3(u, v, w) &= u\sin(v) \end{aligned}$$

Challenges for other users

Reuse and adjust

- JSXGraph code grows rapidly
- Idea: decouple math (question variables) and visualization
- Documentation

1. Integration 2D
2. Integration 3D
3. **Calculus of 2D functions**
4. Vector Fields

Calculus of 2D functions

Mathematical focus

- necessary condition of 1. order (Tangential plane)
- sufficient condition of 2. order
- local approximation is given by

$$T_1(u) = f(u_0) + \langle \nabla f(u_0), u - u_0 \rangle$$

and

$$T_2(u) = f(u_0) + \langle \nabla f(u_0), u - u_0 \rangle + (u - u_0)^\top H_f(u_0)(u - u_0)$$

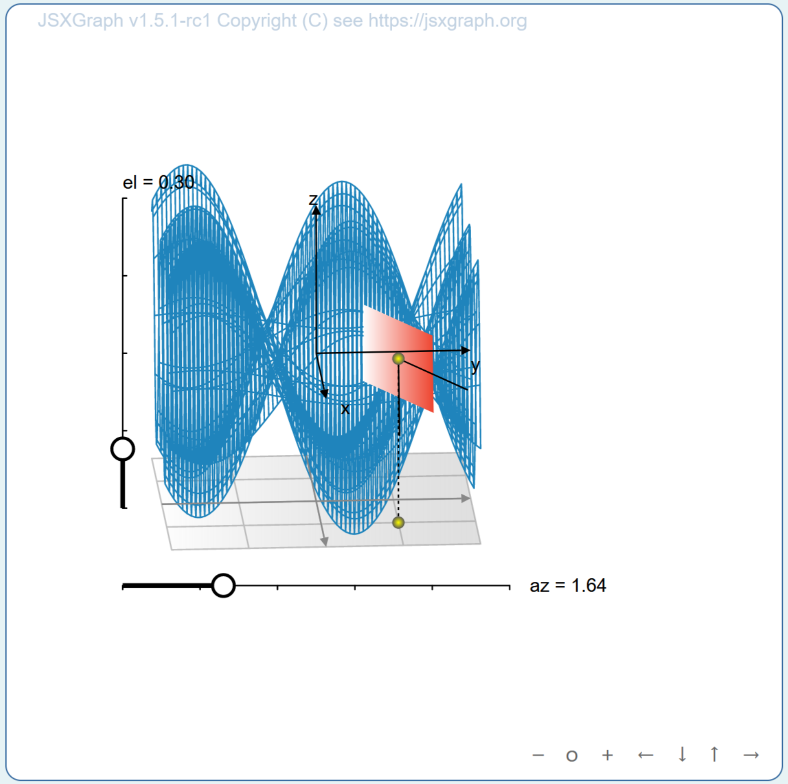
Aim

- Student knows the concept of stationary points in 2D and can calculate them (Handling mathematical symbols and formalism)
- Student understands, how stationary points and tangent planes are connected graphically (Representing mathematical entities)
- Using a visualization of planes tangent to a 2D function graph student can decide whether or not a point is stationary (Making use of aids and tools)

First Order condition

Given is the graph of a function and its tangent plane in a point. You can move that point by moving its projected point in x, y-direction. You can rotate the coordinate system using the sliders.

Select a stationary point of the function.



First Order condition

Example function

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \mathbf{x} \mapsto a_1 \cos(a_2 \pi x_1) \cos(a_3 x_2)$$

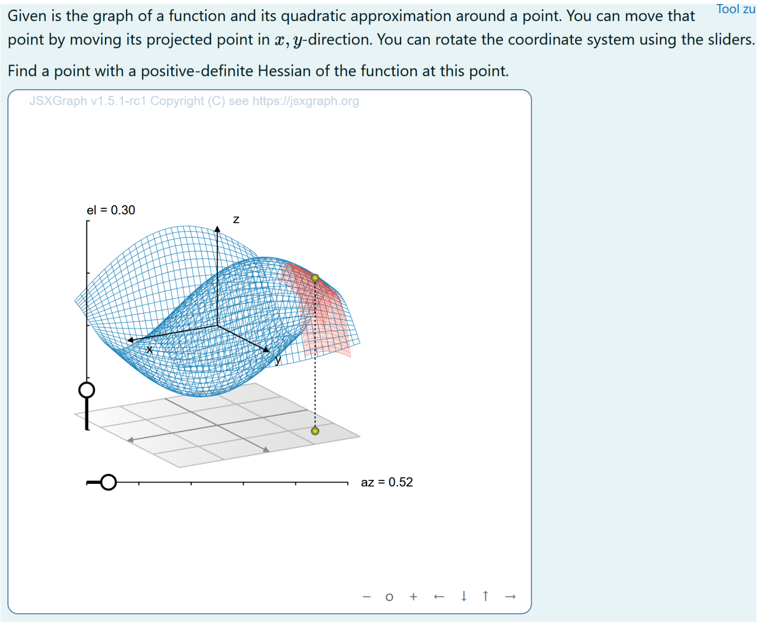
Focus in implementation

- random values for parameters in function
- transfer function from question variables to JSXGraph

```
F:a1 * cos(%pi*a2 * x )* cos( a3 * y);
```

- attach local derivative at the graph
- reusability / easy to adopt

Second Order condition




```
// Maxima
F:a1 * cos(%pi*a2 * x )* cos( a3 * y);
Fdx:diff(F,x);
Fdy:diff(F,y);
Fdxx:diff(Fdx,x);
Fdxy:diff(Fdy,x);
Fdyy:diff(Fdy,y);
// JSXGraph, JessieCode
var F = board.jc.snippet('#F#', true, 'x,y');
var Fdx = board.jc.snippet('#Fdx#', true, 'x,y');
var Fdx = board.jc.snippet('#Fdy#', true, 'x,y');
var Fdxx = board.jc.snippet('#Fdxx#', true, 'x,y');
var Fdxy = board.jc.snippet('#Fdxy#', true, 'x,y');
var Fdyy = board.jc.snippet('#Fdyy#', true, 'x,y');
```

Calculus of 2D functions

Implementation

- Not easy to find useful combination of parameters and functions
- Binding for `STACK` and `point3D` not clear
  need for the option **Check**
- Goal: User (teacher) has only to adopt the function and `rand` commands in field `question variables`

Curl of a vector field

Idea

- Make vector fields visible
- practice the calculation of a curl
- ideal to create an adaptive (tutorial) task

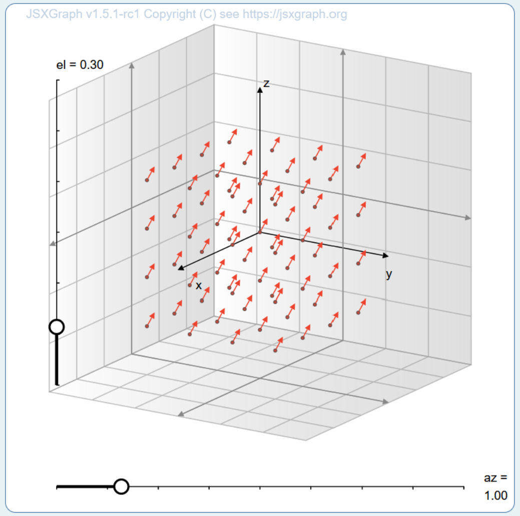
Aim

- Student knows how to calculate partial derivatives, vector products and curls of 3D vector fields (Handling mathematical symbols and formalism)
- Student understands, how a vector field and its curl are connected graphically (Representing mathematical entities)
- Using a visualization of vector field and its curl the student can graphically check whether his calculations are correct (Making use of aids and tools)

Curl of a vector field

Given is the curl of a vector field $\hat{V}: \mathbb{R}^3 \to \mathbb{R}^3$ by $\hat{V}(x,y,z) := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ as shown in the diagram.

Select the vector field V, so that $\hat{V} = \nabla \times V$ is valid.

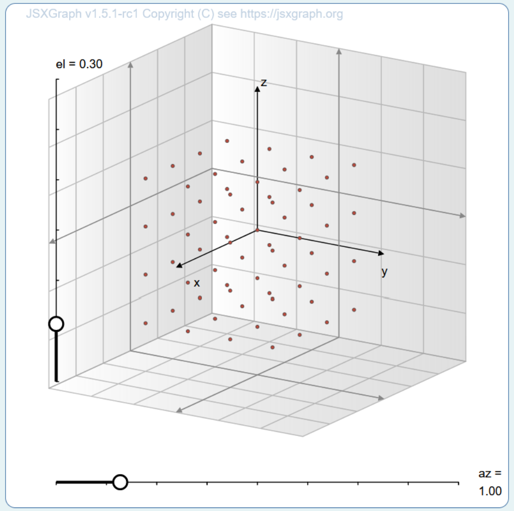


Select V.

- (Nicht beantwortet)
- $[z, x, y]$
- $[y, -x, 0]$
- $[x, y, z]$

Given is the curl of a vector field $\hat{V}: \mathbb{R}^3 \to \mathbb{R}^3$ by $\hat{V}(x,y,z) := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ as shown in the diagram.

Select the vector field V, so that $\hat{V} = \nabla \times V$ is valid.



Select V.

- (Nicht beantwortet)
- $[z, x, y]$
- $[y, -x, 0]$
- $[x, y, z]$

Teacher‘s perspective

Question variables

Question text

JSXGraph Code
- setup of coordinate system
- plotting
- binding to answer

Potential response tree

Student‘s perspective

Introductory text

JSXGraph applet
- 3D objects defined in code
- values, angles can be read or estimated

Input of answer

Feedback